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UNIQUE PRIMITIVE PYTHAGOREAN TRIPLES FOR EVERY INTEGER AND FOR EVERY SET OF TWO INTEGERS

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ABSTRACT

The study of Pythagorean triples is very old, and may possibly predate Pythagoras. One of the numerous related concepts that is studied are formulae to generate primitive Pythagorean triples. A formula given by Euclid requires an input of two parameters, herein called M and N. Euclid's formula has the "advantages" of (1) being unique, meaning any change in either M or N will change the triple that results and (2) being exhaustive, meaning all primitive Pythagorean triples may be generated by Euclid's formula. However, it has the "disadvantage" that there is a dependency between M and N, and they cannot be chosen independently. In this paper we present two alternate formulae to generate primitive Pythagorean triples that do not have this disadvantage. The first only requires one parameter which may be any positive integer, and the second requires an input of two parameters that are totally independent, and may be any positive integers. These two formulae are also unique, but they are not exhaustive—there exist primitive Pythagorean triples that are not generated by either the first or second of these formulae.

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1. PYTHAGOREAN TRIPLES: HISTORY, NOTATION, DEFINITIONS, BASIC FACTS

1.1 History

Perhaps the most famous theorem of all of mathematics is the Pythagorean Theorem. There are over three hundred different proofs of this theorem [4], perhaps more than for any other theorem. Even President Garfield devised a new proof, which is believed to be the only original contribution to mathematics made by a U. S. President [2]. The study of the Pythagorean Theorem and related problems started well before the time of Pythagoras in many major ancient civilizations [1], [6] and continues until this day. The Pythagorean Theorem, and the attempt to solve many related problems, have spawned much growth and development of modern algebra. Perhaps the most famous of the related problems—and perhaps the most famous and longest unsolved problem in all of mathematics—is Fermat's Last Theorem, which was first conjectured by Pierre de Fermat in 1637, dubbed by the Guinness Book of World Records as the "most difficult math problem," [3] and finally solved by Andrew Wiles of Princeton in 1995.

1.2 Notation

Throughout this paper, all variables refer to positive integers.

1.3 Definitions and Basic Facts

a. A right-angle triangle, or right triangle is a triangle in which two sides meet at a right angle, an angle of ninety degrees or $\pi/2$ radians. The two sides forming the right angle are called arms or legs of the triangle, and the side opposite the right angle is called the hypotenuse.

b. The Pythagorean Theorem states if A, B, and C are the lengths of the sides of a right triangle, and C is the length of the hypotenuse, then $A^2 + B^2 = C^2$.

c. A Pythagorean triple is a set of any three positive integers, (A, B, C), such that $A^2 + B^2 = C^2$. (Note by requiring A and B to be positive, we exclude the trivial case when A or B = 0.) Famous examples are (3, 4, 5), (5, 12, 13), and multiples of them, such as (6, 8, 10) or (30, 40, 50).

d. There are positive real numbers that are not integers, that satisfy the equation, $A^2 + B^2 = C^2$, such as (3/5, 4/5, 1) or (1, 1, $\sqrt{2}$) However, they are not designated as "Pythagorean triples."

e. A Pythagorean triple is said to be primitive if there is no positive integer greater than one that divides each of A, B, or C. Since a divisor of any two of the equation $X + Y = Z$ divides the third, the condition of dividing just two of the three of A, B, C is equivalent to dividing all three, so it follows that if any number divides just two of the three A, B, C, then the triple is not primitive.

f. Two Pythagorean triples, T_1 and T_2, where $T_1 = (A_1, B_1, C_1)$ and $T_2 = (A_2, B_2, C_2)$ are said to be equivalent, written as $T_1 \sim T_2$ if and only if there is some triple, $T_0 = (A_0, B_0, C_0)$, not necessarily distinct from T_1 or T_2, and two positive integers m_1 and m_2, such that $(A_1, B_1, C_1) = (m_1A_0, m_1B_0, m_1C_0)$ and $(A_2, B_2, C_2) = (m_2A_0, m_2B_0, m_2C_0)$. For example, if T_1 = (18, 24, 30) and T_2 = (54, 72, 90), then $T_1 \sim T_2$, because T_1 can be considered 3 times (6, 8, 10) and T_2 can be considered 9 times (6, 8, 10). Note however, they can also be considered 6 times, and 18 times (respectively) the Pythagorean triple (3, 4, 5). Note further that neither T_1 nor T_2 need be a multiple of the other.

g. All Pythagorean triples equivalent to one given triple forms an equivalence class, that is to say:

If T_1, T_2, and T_3 are any three Pythagorean triples, then the relation "~" satisfies these three conditions:

(1) Reflexive: $T_1 \sim T_1$

(2) Symmetric: $T_1 \sim T_2 \Rightarrow T_2 \sim T_1$

(3) Transitive: $(T_1 \sim T_2)$ & $(T_2 \sim T_3) \Rightarrow (T_1 \sim T_3)$

It then follows that all triples equivalent to (3, 4, 5) form one equivalence class and all triples equivalent to (5, 12, 13) form another separate equivalence class, because (3, 4, 5) is not equivalent to (5, 12, 13).

h. Using the concept of equivalence, a primitive Pythagorean triple is one where all others in the class are multiples of it. Hence, every equivalence class of Pythagorean triples contains only one primitive triple, and all other triples are multiples of the primitive one. For example, even though (18, 24, 30) is a multiple of (6, 8, 10), the triple (6, 8, 10) is not primitive because there is at least one member of this class, namely (3, 4, 5), that is not a multiple of (6, 8, 10); however all members of this class are multiples of (3, 4, 5), so that is the one and only primitive member of this class.

2. EUCLID'S FORMULA FOR PRIMITIVE PYTHAGOREAN TRIPLES

(A, B, C) is a primitive Pythagorean triple if and only if there exist two positive integers, M and N, satisfying three conditions:

(1) $M > N > 0$

(2) M and N are relatively prime

(3) M and N are of opposite parity (one is odd and the other is even),

where

$A = M^2 - N^2$

$B = 2MN$

$C = M^2 + N^2$

These last three equations are called "Euclid's formula for primitive Pythagorean triples[i]." In Appendix A we provide a proof of the uniqueness of Euclid's formula, meaning that every Pythagorean triple can be generated by only one unique pair of M and N. Any change in either M or N will change the triple that results. In Appendix B we discuss the effects of violating any of the above conditions.

In the general solution, M and N cannot be chosen independently, since M and N depend on each other. For instance, if M is odd, then N must be even.

In this paper we are looking for solutions in which we can choose **any** one integer as an input parameter, or **any** set of two integers as input parameters, to generate a primitive Pythagorean triple. Furthermore, we want all our equations to present **unique** triples.

We present two separate formulae for generating primitive Pythagorean triples, one using any one positive integer as an input, and another one using any two positive integers. Every one of the solutions is a unique Pythagorean triple. However, our formulae do **not** generate all the primitive triples.

2.1 First Formula

For our first formula, we set:

$N = I$

$M = N + 1$

where I is any positive integer.

It is trivial to prove that these equations satisfy the three conditions of Euclid's formula, and they thus produce unique primitive Pythagorean triples.

For this formula,

$A = 2I + 1$

$B = 2I^2 + 2I$

$C = 2I^2 + 2I + 1$

Furthermore,

$C – A = 2N^2 = 2I^2$

$(C – B)^{1/2} = M - N = 1$

2.2 Second Formula

For our second formula, we set[ii]:

$N = 2I\text{-}1$

$M = N + 2J + 1$

where I and J are any two positive integers.

Again, it is simple to prove that these equations satisfy the three conditions of Euclid’s formula, and they thus produce unique primitive Pythagorean triples. Since I and J are positive, M > N > 0. If I = 1, N = 1, which is relatively prime with any number and M = 2(J + 1), so M and N are of opposite parity. If I > 1, N is a power of 2 and M is odd, so they are relatively prime and of opposite parity. Therefore, this formula produces primitive Pythagorean triples which are all unique.

For this equation,

$A = (2^{I-1} + 2J + 1)^2 – 2^{2I-2}$

$B = 2^I (2^{I-1} + 2J + 1)^2$

$C = (2^{I-1} + 2J + 1)^2 + 2^{2I-2}$

Furthermore,

$C – A = 2N^2 = 2^{2I-1}$

$(C – B)^{1/2} = M - N = 2J + 1$

A simple proof that all triples produced by Equation 1 are different than all triples produced by Equation2 follows. For Equation 1, $(C – B)^{1/2} = 1$. For Equation 2, $(C – B)^{1/2} = J + 1$ which must be greater or equal to 2, since the minimum value of J = 1.

Table 1 presents a summary of the two equations and their relationship to Euclid’s general solution.

Table 1. Summary of the Two Equations

	General Solution	**First Equation**	**Second Equation**
N		I	2^{I-1}
M		N + 1	N + 2J +1
A	$M^2 - N^2$	2I + 1	$(2^{I-1} + 2J + 1)^2 - 2^{2I-2}$
B	2MN	$2I^2 + 2I$	$2^I (2^{I-1} + 2J + 1)$
C	$M^{2+} N^2$	$2I^2 + 2I + 1$	$(2^{I-1} + 2J + 1)^2 + 2^{2I-2}$
C- A	$2N^2$	$2I^2$	2^{2I-1}
$(C-B)^{1/2}$	M – N	1	2J + 1

Appendix A: Proof of the Uniqueness of Euclid's Formula

Statement of theorem: If two pairs of positive integers (M, N) and (m, n), in which both pairs satisfy conditions (1) M > N > 0, (2) M and N are relatively prime and (3) M and N are of opposite parity generate the same primitive Pythagorean triple, (A, B, C), then m = M and n = N.

Proof: Indirect proof. We assume there are two distinct pairs (M, N) and (m, n) that produce the same triple (A, B, C) and get a contradiction.

From Euclid's formula, $M^2 = (A + C)/2$ and $N^2 = (C – A)/2$. By the same token, $m^2 = (A + C)/2$ and $n^2 = (C – A)/2$. Therefore $M^2 = m^2$ and $N^2 = n^2$. Since all four numbers, M, N, m, n are positive, this means M = m, and N = n, contradicting the assumption that the two pairs (M, N) and (m, n) are distinct.

Appendix B: Violating of the Conditions for Euclid's Formula

For **any** real values of M and N, the numbers A, B, C, given by Euclid's formula, will still satisfy the Pythagorean equation of $A^2 + B^2 = C^2$, but (A, B, C) may not necessarily be a Pythagorean triple, which, by definition, means A, B, and C are all **positive integers**. Thus, when M and N are real numbers, but they are not both integers, no Pythagorean triples will be produced, even though $A^2 + B^2 = C^2$.

If M and N are both integers, but any of the three conditions listed above are not met, there are different possibilities:

(1) If the first condition is violated:

(1.1) If either M or N is zero (or both), then B = 0 and A = C, so (A, B, C) is not a Pythagorean triple.

(1.2) If M = N then A = 0 and B = C, so (A, B, C) is not a Pythagorean triple.

(1.3) If either M or N is negative, but not both, then B is negative, and again, (A, B, C) is not a Pythagorean triple, but (A, |B|, C) is.

(1.4) If both M and N are negative, then (A, B, C) is a Pythagorean triple that is identical to the triple where both M and N are positive.

(2) If the second condition is violated, M and N are integers but not relatively prime, so they are divisible by a common factor, and A, B and C are divisible by the square of that factor, meaning (A, B, C) is a Pythagorean triple, but it is not primitive.

(3) If the third condition is violated, M and N are both odd or both even, then (A, B, C) is a Pythagorean triple, but all of A, B, and C are even, so again the Pythagorean triple is not primitive.

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[i]There are formulae other than Euclid's that produce all primitive Pythagorean triples. For alternative formulations, see [5], [7].

[ii] If we define M = N + 2I -1, then some of the triples from our second equation will duplicate triples in our first equation.